



The Open
University

MST124

Essential mathematics 1

Exercise Booklet 12

1 Arithmetic with complex numbers

Exercise 1

Write down the real and imaginary parts of each of the following complex numbers.

- (a) $\sqrt{3} + 4i$ (b) 2.5 (c) $17i$

Exercise 2

Work out $z + w$ and $z - w$ for each of the following cases.

- (a) $z = 4 + 2i$, $w = 6 + 5i$
 (b) $z = -2$, $w = 3 + i$
 (c) $z = -4.1 + 3.2i$, $w = 2.0 - 6.7i$
 (d) $z = 4 + 5i$, $w = 6 - 5i$
 (e) $z = \frac{4}{3}i$, $w = \frac{3}{5} + 2i$

Exercise 3

Let $u = 3 + 2i$, $v = -4 + 3i$ and $w = 1 - 5i$.

- (a) Work out the following.
 (i) $u + v + w$ (ii) $u + v - w$
 (iii) $u - v + w$
 (b) Use one of your answers to part (a) to work out $w - (u + v)$.

Exercise 4

Let $p = 1 + 2i$, $q = 2 - i$, $r = -\frac{1}{2} + 4i$, $s = 5i$ and $t = -3i$.

Find the following products.

- (a) pq (b) qr (c) rs (d) st
 (e) tp (f) pr

Exercise 5

Let $u = -3 + 2i$, $v = 2 - i$ and $w = 5i$.

Work out the following.

- (a) $(u + v)w$ (b) $uv + w$ (c) uvw

Exercise 6

- (a) Work out 2^4 , $(-2)^4$, $(2i)^4$ and $(-2i)^4$.
 (b) What is the connection between 2 , -2 , $2i$ and $-2i$?

Exercise 7

Find the complex conjugate of each of the following complex numbers, giving your answer in the form $a + bi$. If necessary, simplify the number first.

- (a) $3 - 4i$ (b) $-5i$ (c) $3(4 - 2i)$
 (d) $2i(5 + 3i)$

Exercise 8

Multiply each of the following complex numbers by its complex conjugate.

- (a) $5 + 2i$ (b) $-2 + 3i$ (c) $3 - 5i$
 (d) $3i$ (e) 5

Exercise 9

Write each of the following fractions as a single complex number.

- (a) $\frac{3 - 2i}{4 + 3i}$ (b) $\frac{-4 + 3i}{-i}$ (c) $\frac{3 - 4i}{2 - i}$
 (d) $\frac{2}{3 + i}$ (e) $\frac{3i}{2 - 5i}$ (f) $\frac{2 + 3i}{4i}$

2 Geometry with complex numbers

Exercise 10

Mark the following complex numbers on a diagram of the complex plane:

$$3 + 2i, \quad 3 - 2i, \quad -2 + 3i, \quad -4i, \quad -3.$$

Exercise 11

- (a) Given that $z = 1 + 2i$ and $w = 3 + i$, use geometric methods to mark $z + w$, $z - w$, $\overline{z + w}$ and $-2z$ on a diagram of the complex plane.
- (b) Confirm your answer to part (a) by working out each complex number algebraically in the form $a + bi$.

Exercise 12

Find the modulus of each of the following complex numbers.

- (a) $2 + 4i$ (b) 6 (c) $8i$
 (d) $-2 - 7i$ (e) $3 - i\sqrt{2}$

Exercise 13

Find the reciprocal of each of the following complex numbers.

- (a) $3 + 2i$ (b) $2 - i$ (c) $-4i$

Exercise 14

Find the principal argument of each of the following complex numbers.

- (a) $2i$ (b) $-2i$ (c) -2 (d) 3

Exercise 15

Find the principal argument of each of the following complex numbers.

- (a) $1 - i$ (b) $3(-1 + i)$
 (c) $\sqrt{3} + i$ (d) $-2 - 2\sqrt{3}i$

Exercise 16

Write the following complex numbers in Cartesian form.

- (a) $2 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right)$
 (b) $3 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right)$

- (c) $4 \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right)$
 (d) $3(\cos \pi + i \sin \pi)$
 (e) $\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right)$

Exercise 17

Use your solutions to Exercises 14 and 15 to write the following complex numbers in polar form.

- (a) $2i$ (b) $-2i$ (c) -2 (d) 3
 (e) $1 - i$ (f) $3(-1 + i)$ (g) $\sqrt{3} + i$
 (h) $-2 - 2\sqrt{3}i$

Exercise 18

Find zw in polar form in each of the following cases. Then write your answers in Cartesian form.

- (a) $z = 3 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right),$
 $w = 2 \left(\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) \right)$
 (b) $z = \cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right),$
 $w = \cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)$

Exercise 19

Let

$$u = 2 \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right)$$

$$v = 3 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right),$$

and

$$w = 4 \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right).$$

Find the following products in polar form.

- (a) uv (b) uw (c) uvw

Exercise 20

Let

$$u = 10 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right),$$

$$v = 5 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right)$$

and

$$w = 2 \left(\cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right) \right).$$

- (a) Find the following quotients in polar form.

(i) u/v (ii) v/w (iii) u/w

- (b) Use your answers to part (a) to confirm that

$$\frac{u}{v} \times \frac{v}{w} = \frac{u}{w}.$$

Exercise 21

- (a) Describe the geometric effect of multiplying a complex number by $-3i$.
- (b) Use polar form to show that the geometric effect of multiplying a complex number by -2 is a rotation through a half turn and a scaling by a factor of 2.

Exercise 22

Work out the Cartesian forms of the following complex numbers, using de Moivre's formula if appropriate.

(a) $\left(2 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right) \right)^3$

(b) $\left(3^{1/4} \left(\cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right) \right) \right)^4$

(c) $(-1 + i)^4$ (d) $(-1 + i)^{-4}$

(e) $(3\sqrt{3} + 9i)^{-5}$

3 Polynomial equations

Exercise 23

Solve the following quadratic equations by completing the square where necessary.

(a) $z^2 + 2z + 3 = 0$

(b) $z^2 - 3z + 13 = 0$

(c) $z^2 + 36 = 0$

Exercise 24

Solve the following quadratic equations by using the quadratic formula.

(a) $z^2 - 2z + 3 = 0$

(b) $z^2 + 3z + 13 = 0$

(c) $2z^2 + 6z + 5 = 0$

Exercise 25

Find, in their simplest forms, quadratic equations with the following pairs of solutions.

(a) $2 \pm 3i$ (b) $-3 \pm 2i$ (c) $4 \pm \frac{1}{2}i$

(d) $\pm 6i$

Exercise 26

Solve the equation $z^8 = 1$. Give your answers in both polar form and Cartesian form. Sketch the solutions in the complex plane.

Exercise 27

Solve the following equations. Give your answers in both polar form and Cartesian form. Sketch the solutions in the complex plane.

(a) $z^4 = -16$ (b) $z^3 = 27i$

4 Exponential form

Exercise 28

Write the following complex numbers in Cartesian form.

- (a) $5e^{i\pi/2}$ (b) $2e^{-i\pi/6}$ (c) $2e^{3\pi i/4}$
-

Exercise 29

Write the following complex numbers in exponential form.

- (a) 3 (b) $\frac{1}{2}i$ (c) $-\frac{3}{2}i$
 (d) $-3 - i\sqrt{3}$
-

Exercise 30

Express each of the following products, quotients and powers of complex numbers as a single complex number in exponential form and Cartesian form.

- (a) $e^{i\pi/6} \times e^{i\pi/2}$ (b) $(e^{i\pi/8})^2$
 (c) $\frac{e^{i\pi/2}}{e^{i\pi/3}}$ (d) $2e^{i\pi/3} \times 3e^{-i\pi/3}$
 (e) $(4e^{i\pi})^3$
-

Exercise 31

Use Euler's formula to verify the equation

$$e^{-i\pi/2} + i = 0.$$

Exercise 32

Use de Moivre's formula to obtain the trigonometric identities

$$\begin{aligned}\cos 5\theta &= 5 \cos \theta - 20 \cos^3 \theta + 16 \cos^5 \theta \\ \sin 5\theta &= 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta.\end{aligned}$$

To condense your working, you may find it helpful to write $c = \cos \theta$ and $s = \sin \theta$.

Exercise 33

Use the formulas

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

to obtain the following identities.

- (a) $\sin^2 \theta + \cos^2 \theta = 1$
 (b) $\cos^2 \theta - \sin^2 \theta = \cos 2\theta$
 (c) $2 \sin \theta \cos \theta = \sin 2\theta$
-

Solutions to exercises

Solution to Exercise 1

$$(a) \operatorname{Re}(\sqrt{3} + 4i) = \sqrt{3}, \quad \operatorname{Im}(\sqrt{3} + 4i) = 4$$

$$(b) \operatorname{Re}(2.5) = 2.5, \quad \operatorname{Im}(2.5) = 0$$

$$(c) \operatorname{Re}(17i) = 0, \quad \operatorname{Im}(17i) = 17$$

(Note that the imaginary part of a complex number is always a real number.)

Solution to Exercise 2

$$\begin{aligned} (a) \quad z + w &= (4 + 2i) + (6 + 5i) \\ &= (4 + 6) + (2 + 5)i \\ &= 10 + 7i \end{aligned}$$

$$\begin{aligned} z - w &= (4 + 2i) - (6 + 5i) \\ &= (4 - 6) + (2 - 5)i \\ &= -2 - 3i \end{aligned}$$

$$\begin{aligned} (b) \quad z + w &= (-2) + (3 + i) \\ &= (-2 + 3) + (0 + 1)i \\ &= 1 + i \end{aligned}$$

$$\begin{aligned} z - w &= (-2) - (3 + i) \\ &= (-2 - 3) + (0 - 1)i \\ &= -5 - i \end{aligned}$$

$$\begin{aligned} (c) \quad z + w &= (-4.1 + 3.2i) + (2.0 - 6.7i) \\ &= (-4.1 + 2.0) + (3.2 + (-6.7))i \\ &= -2.1 - 3.5i \end{aligned}$$

$$\begin{aligned} z - w &= (-4.1 + 3.2i) - (2.0 - 6.7i) \\ &= (-4.1 - 2.0) + (3.2 - (-6.7))i \\ &= -6.1 + 9.9i \end{aligned}$$

$$\begin{aligned} (d) \quad z + w &= (4 + 5i) + (6 - 5i) \\ &= (4 + 6) + (5 + (-5))i \\ &= 10 \end{aligned}$$

$$\begin{aligned} z - w &= (4 + 5i) - (6 - 5i) \\ &= (4 - 6) + (5 - (-5))i \\ &= -2 + 10i \end{aligned}$$

$$\begin{aligned} (e) \quad z + w &= \left(\frac{4}{3}i\right) + \left(\frac{3}{5} + 2i\right) \\ &= \left(0 + \frac{3}{5}\right) + \left(\frac{4}{3} + 2\right)i \\ &= \frac{3}{5} + \frac{10}{3}i \end{aligned}$$

$$\begin{aligned} z - w &= \left(\frac{4}{3}i\right) - \left(\frac{3}{5} + 2i\right) \\ &= \left(0 - \frac{3}{5}\right) + \left(\frac{4}{3} - 2\right)i \\ &= -\frac{3}{5} - \frac{2}{3}i \end{aligned}$$

Solution to Exercise 3

$$\begin{aligned} (a) \quad (i) \quad u + v + w &= (3 + 2i) + (-4 + 3i) + (1 - 5i) \\ &= (3 + (-4) + 1) + (2 + 3 + (-5))i \\ &= 0 \end{aligned}$$

$$\begin{aligned} (ii) \quad u + v - w &= (3 + 2i) + (-4 + 3i) - (1 - 5i) \\ &= (3 + (-4) - 1) + (2 + 3 - (-5))i \\ &= -2 + 10i \end{aligned}$$

$$\begin{aligned} (iii) \quad u - v + w &= (3 + 2i) - (-4 + 3i) + (1 - 5i) \\ &= (3 - (-4) + 1) + (2 - 3 + (-5))i \\ &= 8 - 6i \end{aligned}$$

$$\begin{aligned} (b) \quad w - (u + v) &= w - u - v \\ &= -u - v + w \\ &= -(u + v - w) \end{aligned}$$

From part (b)(ii),

$$\begin{aligned} -(u + v - w) &= -(-2 + 10i) \\ &= 2 - 10i. \end{aligned}$$

Solution to Exercise 4

$$\begin{aligned} (a) \quad pq &= (1 + 2i)(2 - i) \\ &= 2 - i + 4i - 2i^2 \\ &= 2 - i + 4i + 2 \\ &= 4 + 3i \end{aligned}$$

$$\begin{aligned} (b) \quad qr &= (2 - i)\left(-\frac{1}{2} + 4i\right) \\ &= -1 + 8i + \frac{1}{2}i - 4i^2 \\ &= -1 + 8i + \frac{1}{2}i + 4 \\ &= 3 + \frac{17}{2}i \end{aligned}$$

$$\begin{aligned} (c) \quad rs &= \left(-\frac{1}{2} + 4i\right)(5i) \\ &= -\frac{5}{2}i + 20i^2 \\ &= -20 - \frac{5}{2}i \end{aligned}$$

$$\begin{aligned} (d) \quad st &= 5i \times (-3i) \\ &= -15i^2 \\ &= 15 \end{aligned}$$

$$\begin{aligned} (e) \quad tp &= -3i(1 + 2i) \\ &= -3i - 6i^2 \\ &= 6 - 3i \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad pr &= (1 + 2i)(-\tfrac{1}{2} + 4i) \\
 &= -\tfrac{1}{2} + 4i - i + 8i^2 \\
 &= -\tfrac{1}{2} + 4i - i - 8 \\
 &= -\tfrac{17}{2} + 3i
 \end{aligned}$$

Solution to Exercise 5

$$\begin{aligned}
 \text{(a)} \quad (u + v)w &= ((-3 + 2i) + (2 - i))(5i) \\
 &= (-1 + i)(5i) \\
 &= -5i + 5i^2 \\
 &= -5 - 5i
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad uv + w &= (-3 + 2i)(2 - i) + 5i \\
 &= -6 + 3i + 4i - 2i^2 + 5i \\
 &= -6 + 3i + 4i + 2 + 5i \\
 &= -4 + 12i
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad uvw &= (-3 + 2i)(2 - i)(5i) \\
 &= (-6 + 3i + 4i - 2i^2)(5i) \\
 &= (-4 + 7i)(5i) \\
 &= -20i + 35i^2 \\
 &= -35 - 20i
 \end{aligned}$$

Solution to Exercise 6

$$\begin{aligned}
 \text{(a)} \quad 2^4 &= 16 \\
 (-2)^4 &= (-1)^4 \times 2^4 \\
 &= 1 \times 16 \\
 &= 16
 \end{aligned}$$

$$\begin{aligned}
 (2i)^4 &= 2^4 i^4 \\
 &= 16 \times i^2 \times i^2 \\
 &= 16 \times (-1) \times (-1) \\
 &= 16
 \end{aligned}$$

$$\begin{aligned}
 (-2i)^4 &= (-2)^4 i^4 \\
 &= 16 \times 1 \\
 &= 16
 \end{aligned}$$

(b) 2, -2, 2i and -2i are all fourth roots of 16.

Solution to Exercise 7

(a) $3 + 4i$

(b) $5i$

(c) We have

$$3(4 - 2i) = 12 - 6i.$$

So the complex conjugate is $12 + 6i$.

(d) We have

$$2i(5 + 3i) = 10i + 6i^2 = -6 + 10i.$$

So the complex conjugate is $-6 - 10i$.

Solution to Exercise 8

(a) $(5 + 2i)(5 - 2i) = 5^2 + 2^2 = 29$

(b) $(-2 + 3i)(-2 - 3i) = (-2)^2 + 3^2 = 13$

(c) $(3 - 5i)(3 + 5i) = 3^2 + (-5)^2 = 34$

(d) $3i \times (-3i) = 3^2 = 9$

(e) $5 \times 5 = 25$

Solution to Exercise 9

$$\begin{aligned}
 \text{(a)} \quad \frac{3 - 2i}{4 + 3i} &= \frac{(3 - 2i)(4 - 3i)}{(4 + 3i)(4 - 3i)} \\
 &= \frac{12 - 9i - 8i + 6i^2}{4^2 + 3^2} \\
 &= \frac{6 - 17i}{25}
 \end{aligned}$$

(You can leave this complex number in the form above, or write it as $\frac{6}{25} - \frac{17}{25}i$. Similar comments apply to parts (d), (e) and (f).)

$$\begin{aligned}
 \text{(b)} \quad \frac{-4 + 3i}{-i} &= \frac{(-4 + 3i) \times i}{-i \times i} \\
 &= \frac{-4i + 3i^2}{-i^2} \\
 &= \frac{-3 - 4i}{1} \\
 &= -3 - 4i
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \frac{3 - 4i}{2 - i} &= \frac{(3 - 4i)(2 + i)}{(2 - i)(2 + i)} \\
 &= \frac{6 + 3i - 8i - 4i^2}{2^2 + (-1)^2} \\
 &= \frac{10 - 5i}{5} \\
 &= 2 - i
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \frac{2}{3 + i} &= \frac{2(3 - i)}{(3 + i)(3 - i)} \\
 &= \frac{6 - 2i}{3^2 + 1^2} \\
 &= \frac{6 - 2i}{10} \\
 &= \frac{3 - i}{5}
 \end{aligned}$$

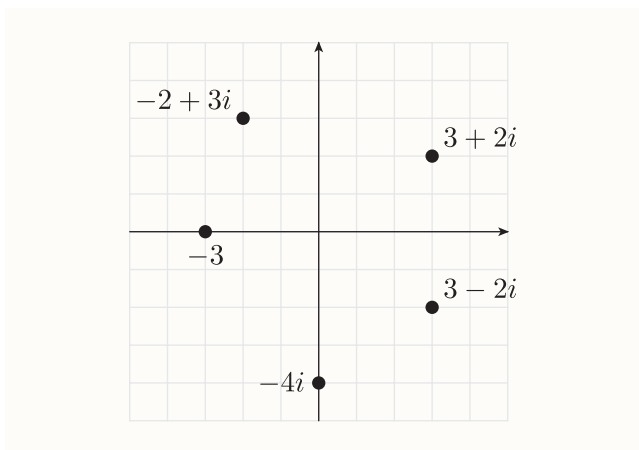
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$$\begin{aligned}
 \text{(e)} \quad \frac{3i}{2-5i} &= \frac{3i(2+5i)}{(2-5i)(2+5i)} \\
 &= \frac{6i+15i^2}{2^2+(-5)^2} \\
 &= \frac{-15+6i}{29}
 \end{aligned}$$

(f) In this case it's simpler to multiply the top and bottom of the fraction by $-i$, to give

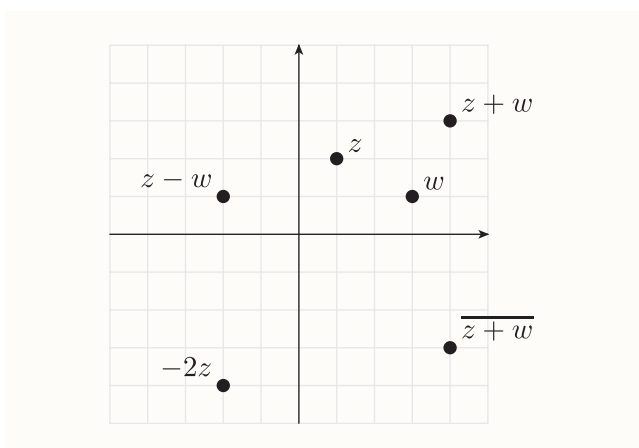
$$\begin{aligned}
 \frac{2+3i}{4i} &= \frac{(2+3i)(-i)}{4i \times (-i)} \\
 &= \frac{-2i-3i^2}{-4i^2} \\
 &= \frac{3-2i}{4}.
 \end{aligned}$$

Solution to Exercise 10



Solution to Exercise 11

(a)



$$\begin{aligned}
 \text{(b)} \quad z+w &= (1+2i) + (3+i) = 4+3i \\
 z-w &= (1+2i) - (3+i) = -2+i \\
 \overline{z+w} &= \overline{4+3i} = 4-3i \\
 -2z &= -2(1+2i) = -2-4i
 \end{aligned}$$

Solution to Exercise 12

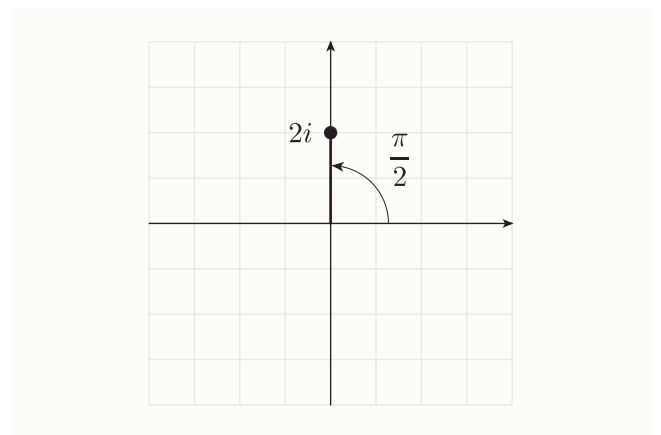
$$\begin{aligned}
 \text{(a)} \quad |2+4i| &= \sqrt{2^2+4^2} = \sqrt{20} = 2\sqrt{5} \\
 \text{(b)} \quad |6| &= 6 \\
 \text{(c)} \quad |8i| &= \sqrt{8^2} = 8 \\
 \text{(d)} \quad |-2-7i| &= \sqrt{(-2)^2+(-7)^2} = \sqrt{53} \\
 \text{(e)} \quad |3-i\sqrt{2}| &= \sqrt{3^2+(-\sqrt{2})^2} = \sqrt{11}
 \end{aligned}$$

Solution to Exercise 13

$$\begin{aligned}
 \text{(a)} \quad \frac{1}{3+2i} &= \frac{\overline{3+2i}}{|3+2i|^2} = \frac{3-2i}{3^2+2^2} = \frac{3-2i}{13} \\
 \text{(b)} \quad \frac{1}{2-i} &= \frac{\overline{2-i}}{|2-i|^2} = \frac{2+i}{2^2+(-1)^2} = \frac{2+i}{5} \\
 \text{(c)} \quad \frac{1}{-4i} &= \frac{1}{-4i} \times \frac{i}{i} = \frac{i}{-4i^2} = \frac{1}{4}i
 \end{aligned}$$

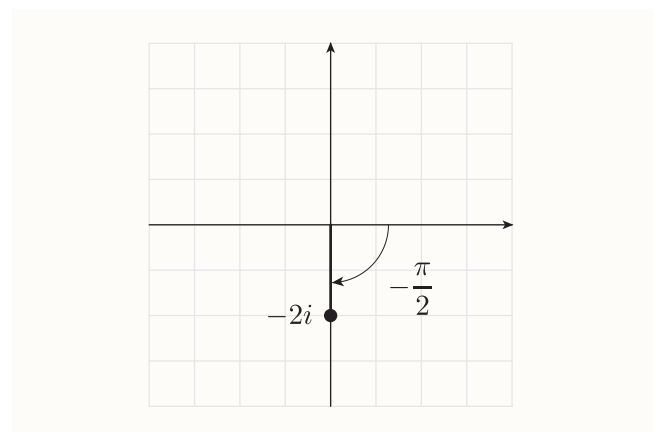
Solution to Exercise 14

(a)



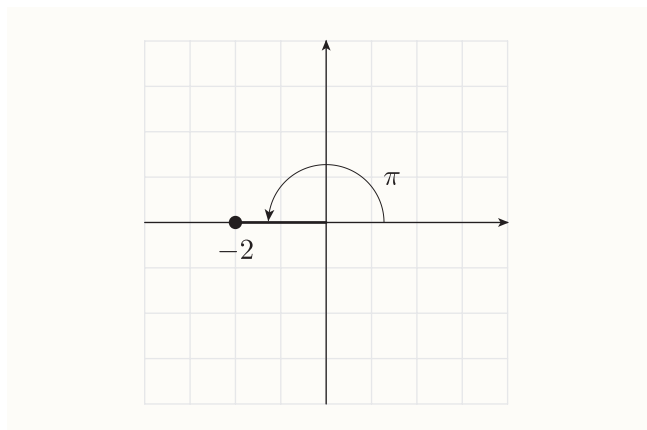
From the diagram, $\text{Arg}(2i) = \pi/2$.

(b)



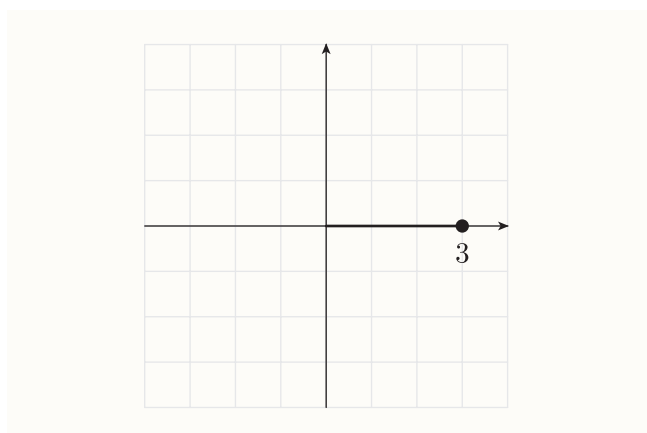
From the diagram, $\text{Arg}(-2i) = -\pi/2$.

(c)



From the diagram, $\text{Arg}(-2) = \pi$.

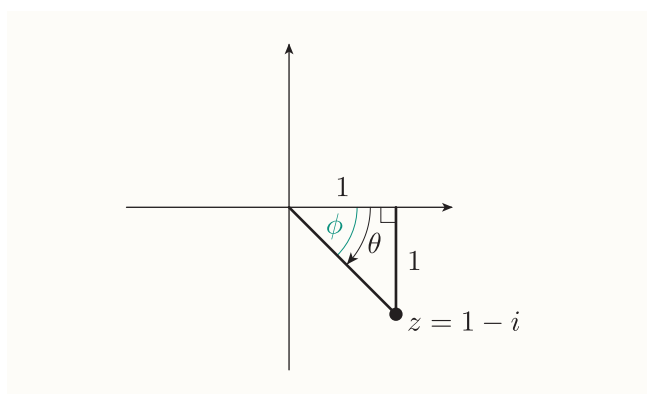
(d)



From the diagram, $\text{Arg}(3) = 0$.

Solution to Exercise 15

(a)



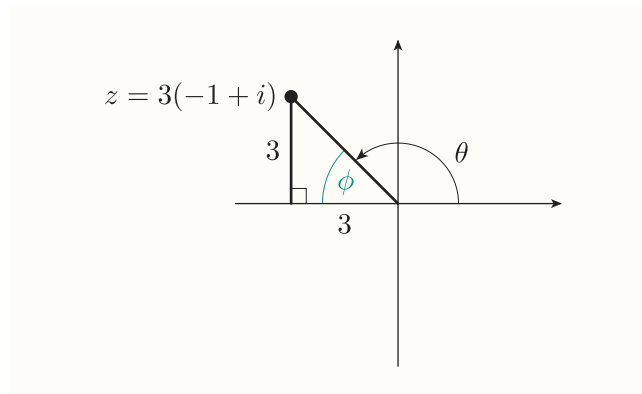
From the diagram,

$$\tan \phi = \frac{1}{1} = 1.$$

Therefore $\phi = \pi/4$. So the principal argument is

$$\theta = -\phi = -\frac{\pi}{4}.$$

(b)



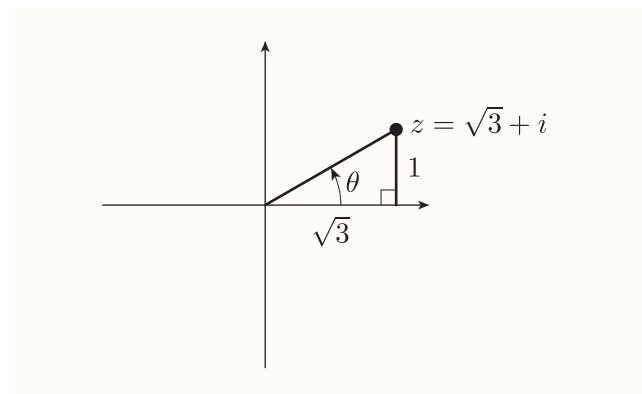
From the diagram,

$$\tan \phi = \frac{3}{3} = 1.$$

Therefore $\phi = \pi/4$. So the principal argument is

$$\theta = \pi - \phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

(c)

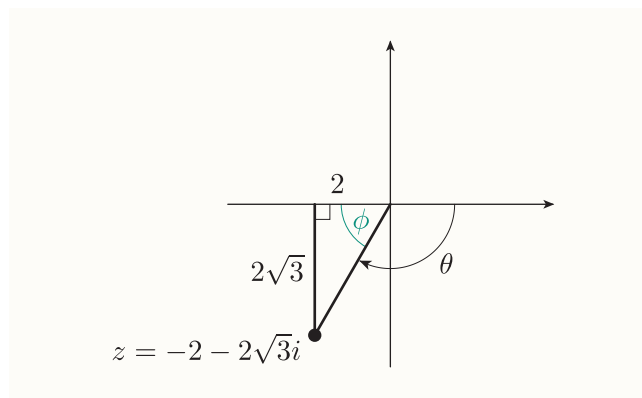


From the diagram,

$$\tan \theta = \frac{1}{\sqrt{3}}.$$

So the principal argument is $\theta = \pi/6$.

(d)



Exercise Booklet 12

From the diagram,

$$\tan \phi = \frac{2\sqrt{3}}{2} = \sqrt{3}.$$

Therefore $\phi = \pi/3$. So the principal argument is

$$\theta = -(\pi - \phi) = -\left(\pi - \frac{\pi}{3}\right) = -\frac{2\pi}{3}.$$

Solution to Exercise 16

$$\begin{aligned} \text{(a)} \quad 2 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right) &= 2 \left(\frac{\sqrt{3}}{2} + \frac{i}{2} \right) \\ &= \sqrt{3} + i \end{aligned}$$

$$\text{(b)} \quad 3 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right) = 3(0 + i) = 3i$$

$$\begin{aligned} \text{(c)} \quad 4 \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right) \\ &= 4 \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right) \\ &= 2\sqrt{2}(-1 + i) \end{aligned}$$

$$\text{(d)} \quad 3(\cos \pi + i \sin \pi) = 3(-1 + 0i) = -3$$

$$\begin{aligned} \text{(e)} \quad \cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) &= \frac{1}{\sqrt{2}} - i \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}}(1 - i) \end{aligned}$$

Solution to Exercise 17

(a) The modulus is 2. From Exercise 14(a), the principal argument is $\pi/2$. Therefore

$$2i = 2 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right).$$

(b) The modulus is 2. From Exercise 14(b), the principal argument is $-\pi/2$. Therefore

$$-2i = 2 \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right) \right).$$

(c) The modulus is 2. From Exercise 14(c), the principal argument is π . Therefore

$$-2 = 2(\cos \pi + i \sin \pi).$$

(d) The modulus is 3. From Exercise 14(d), the principal argument is 0. Therefore

$$3 = 3(\cos 0 + i \sin 0).$$

(e) The modulus is

$$r = \sqrt{1^2 + (-1)^2} = \sqrt{2}.$$

From Exercise 15(a), the principal argument is $-\pi/4$. Therefore

$$1 - i = \sqrt{2} \left(\cos\left(-\frac{\pi}{4}\right) + i \sin\left(-\frac{\pi}{4}\right) \right).$$

(f) The modulus is

$$r = \sqrt{(-3)^2 + 3^2} = \sqrt{18} = 3\sqrt{2}.$$

From Exercise 15(b), the principal argument is $3\pi/4$. Therefore

$$\begin{aligned} &3(-1 + i) \\ &= 3\sqrt{2} \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right). \end{aligned}$$

(g) The modulus is

$$r = \sqrt{(\sqrt{3})^2 + 1^2} = \sqrt{4} = 2.$$

From Exercise 15(c), the principal argument is $\pi/6$. Therefore

$$\sqrt{3} + i = 2 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right).$$

(h) The modulus is

$$\begin{aligned} r &= \sqrt{(-2)^2 + (-2\sqrt{3})^2} \\ &= \sqrt{4 + 4 \times 3} \\ &= \sqrt{16} \\ &= 4. \end{aligned}$$

From Exercise 15(d), the principal argument is $-2\pi/3$. Therefore

$$\begin{aligned} &-2 - 2\sqrt{3}i \\ &= 4 \left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right). \end{aligned}$$

Solution to Exercise 18

(a) In polar form,

$$\begin{aligned} zw &= 3 \times 2 \times \left(\cos\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) + i \sin\left(\frac{\pi}{2} + \frac{5\pi}{6}\right) \right) \\ &= 6 \left(\cos\left(\frac{8\pi}{6}\right) + i \sin\left(\frac{8\pi}{6}\right) \right) \\ &= 6 \left(\cos\left(\frac{4\pi}{3}\right) + i \sin\left(\frac{4\pi}{3}\right) \right). \end{aligned}$$

The angle $4\pi/3$ lies outside the interval $(-\pi, \pi]$, so it isn't the principal argument of zw . The principal argument is given by

$$\frac{4\pi}{3} - 2\pi = \frac{4\pi}{3} - \frac{6\pi}{3} = -\frac{2\pi}{3}.$$

Therefore, in polar form,

$$zw = 6 \left(\cos\left(-\frac{2\pi}{3}\right) + i \sin\left(-\frac{2\pi}{3}\right) \right).$$

In Cartesian form,

$$zw = 6 \left(-\frac{1}{2} - i \frac{\sqrt{3}}{2} \right) = -3 - 3\sqrt{3}i.$$

(b) In polar form,

$$zw = \cos\left(\frac{\pi}{3} + \left(-\frac{\pi}{3}\right)\right) + i \sin\left(\frac{\pi}{3} + \left(-\frac{\pi}{3}\right)\right) \\ = \cos 0 + i \sin 0.$$

In Cartesian form,

$$zw = 1 + 0i = 1.$$

Solution to Exercise 19

$$(a) \quad uv = 3 \times 2 \times \left(\cos\left(\frac{\pi}{4} + \frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{4} + \frac{\pi}{2}\right)\right) \\ = 6 \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right)\right)$$

$$(b) \quad uw = 2 \times 4 \times \left(\cos\left(\frac{\pi}{4} + \frac{3\pi}{4}\right) + i \sin\left(\frac{\pi}{4} + \frac{3\pi}{4}\right)\right) \\ = 8(\cos \pi + i \sin \pi)$$

$$(c) \quad uvw = 2 \times 3 \times 4 \times \left(\cos\left(\frac{\pi}{4} + \frac{\pi}{2} + \frac{3\pi}{4}\right) + i \sin\left(\frac{\pi}{4} + \frac{\pi}{2} + \frac{3\pi}{4}\right)\right) \\ = 24 \left(\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right)\right)$$

The angle $3\pi/2$ lies outside the interval $(-\pi, \pi]$, so it isn't the principal argument of uvw . The principal argument is given by

$$\frac{3\pi}{2} - 2\pi = \frac{3\pi}{2} - \frac{4\pi}{2} = -\frac{\pi}{2}.$$

Therefore

$$uvw = 24 \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right).$$

Solution to Exercise 20

$$(a) \quad (i) \quad \frac{u}{v} = \frac{10}{5} \left(\cos\left(\frac{\pi}{6} - \frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\right) \\ = 2 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right)$$

$$(ii) \quad \frac{v}{w} = \frac{5}{2} \left(\cos\left(\frac{\pi}{3} - \frac{2\pi}{3}\right) + i \sin\left(\frac{\pi}{3} - \frac{2\pi}{3}\right)\right) \\ = \frac{5}{2} \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$$

$$(iii) \quad \frac{u}{w} = \frac{10}{2} \left(\cos\left(\frac{\pi}{6} - \frac{2\pi}{3}\right) + i \sin\left(\frac{\pi}{6} - \frac{2\pi}{3}\right)\right) \\ = 5 \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right)$$

$$(b) \quad \frac{u}{v} \times \frac{v}{w} = 2 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right)\right) \\ \times \frac{5}{2} \left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right) \\ = 5 \left(\cos\left(-\frac{\pi}{6} + \left(-\frac{\pi}{3}\right)\right) + i \sin\left(-\frac{\pi}{6} + \left(-\frac{\pi}{3}\right)\right)\right) \\ = 5 \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right) = \frac{u}{w}$$

Solution to Exercise 21

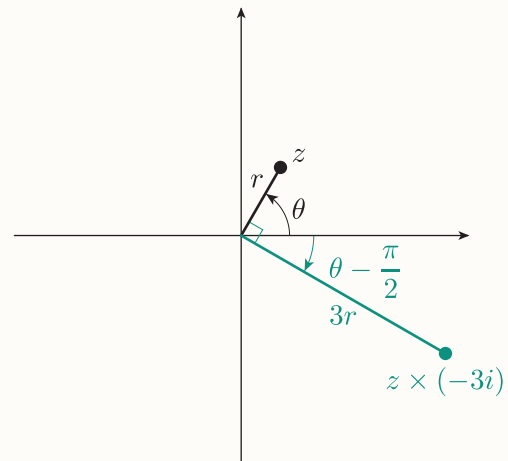
(a) In polar form,

$$-3i = 3 \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right).$$

Let $z = r(\cos \theta + i \sin \theta)$. Then

$$z \times (-3i) = r(\cos \theta + i \sin \theta) \\ \times 3 \left(\cos\left(-\frac{\pi}{2}\right) + i \sin\left(-\frac{\pi}{2}\right)\right) \\ = 3r \left(\cos\left(\theta + \left(-\frac{\pi}{2}\right)\right) + i \sin\left(\theta + \left(-\frac{\pi}{2}\right)\right)\right).$$

Therefore, multiplying z by $-3i$ has the effect of a clockwise rotation through a quarter turn ($-\pi/2$ radians) and a scaling by a factor of 3.



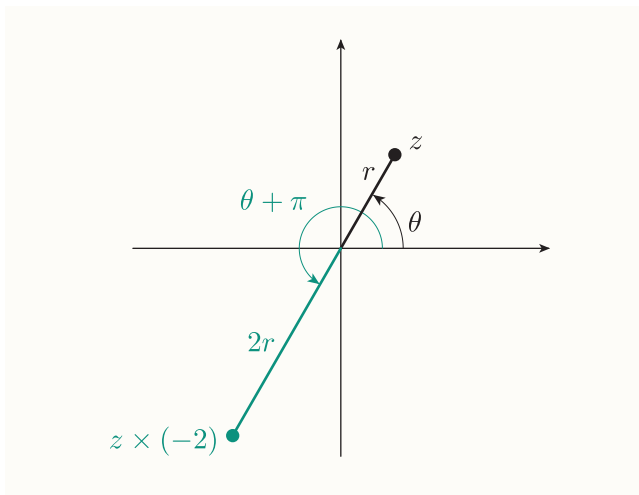
(b) In polar form,

$$-2 = 2(\cos \pi + i \sin \pi).$$

Let $z = r(\cos \theta + i \sin \theta)$. Then

$$z \times (-2) = r(\cos \theta + i \sin \theta) \\ \times 2(\cos \pi + i \sin \pi) \\ = 2r(\cos(\theta + \pi) + i \sin(\theta + \pi)).$$

Therefore, multiplying a complex number by -2 corresponds to a rotation through a half turn (π radians) and a scaling by a factor of 2.



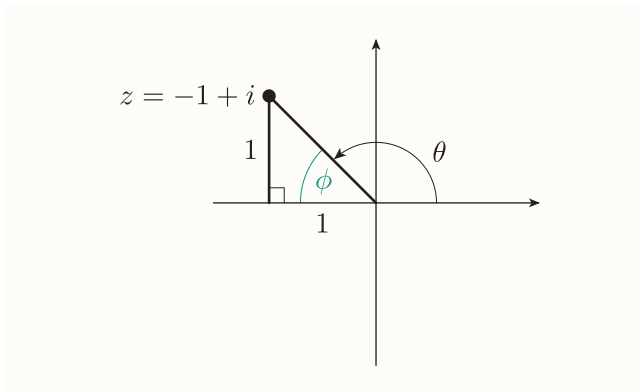
Solution to Exercise 22

$$\begin{aligned}
 \text{(a)} \quad & \left(2 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right) \right)^3 \\
 &= 2^3 \left(\cos\left(3 \times \frac{\pi}{6}\right) + i \sin\left(3 \times \frac{\pi}{6}\right) \right) \\
 &= 8 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right) \\
 &= 8(0 + i) \\
 &= 8i
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \left(3^{1/4} \left(\cos\left(\frac{\pi}{12}\right) + i \sin\left(\frac{\pi}{12}\right) \right) \right)^4 \\
 &= (3^{1/4})^4 \left(\cos\left(4 \times \frac{\pi}{12}\right) + i \sin\left(4 \times \frac{\pi}{12}\right) \right) \\
 &= 3 \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right) \\
 &= 3 \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\
 &= \frac{3}{2}(1 + i\sqrt{3})
 \end{aligned}$$

(c) The modulus of $-1 + i$ is

$$\sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$



From the diagram,

$$\tan \phi = \frac{1}{1} = 1.$$

Therefore $\phi = \pi/4$. So the principal argument is

$$\theta = \pi - \phi = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

In polar form,

$$-1 + i = \sqrt{2} \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right).$$

Hence

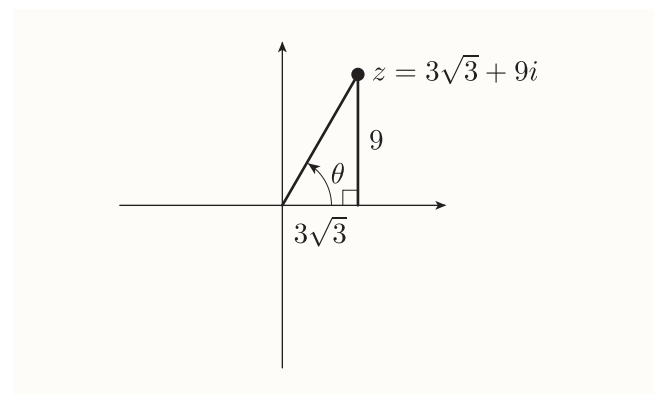
$$\begin{aligned}
 & (-1 + i)^4 \\
 &= (\sqrt{2})^4 \left(\cos\left(4 \times \frac{3\pi}{4}\right) + i \sin\left(4 \times \frac{3\pi}{4}\right) \right) \\
 &= 4 (\cos 3\pi + i \sin 3\pi) \\
 &= 4 (\cos \pi + i \sin \pi) \\
 &= 4(-1 + 0i) \\
 &= -4.
 \end{aligned}$$

(d) De Moivre's formula could be used here, but we can also use the answer to part (c) as follows:

$$(-1 + i)^{-4} = \frac{1}{(-1 + i)^4} = \frac{1}{-4} = -\frac{1}{4}.$$

(e) The modulus of $3\sqrt{3} + 9i$ is

$$\begin{aligned}
 \sqrt{(3\sqrt{3})^2 + 9^2} &= \sqrt{9 \times 3 + 81} \\
 &= \sqrt{108} \\
 &= 6\sqrt{3}.
 \end{aligned}$$



From the diagram,

$$\tan \theta = \frac{9}{3\sqrt{3}} = \sqrt{3}.$$

So the principal argument is $\theta = \pi/3$.

In polar form,

$$3\sqrt{3} + 9i = 6\sqrt{3} \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right).$$

Hence

$$\begin{aligned} & (3\sqrt{3} + 9i)^{-5} \\ &= (6\sqrt{3})^{-5} \left(\cos\left(-5 \times \frac{\pi}{3}\right) + i \sin\left(-5 \times \frac{\pi}{3}\right) \right) \\ &= \frac{1}{(6\sqrt{3})^5} \left(\cos\left(-\frac{5\pi}{3}\right) + i \sin\left(-\frac{5\pi}{3}\right) \right). \end{aligned}$$

The angle $-5\pi/3$ lies outside the interval $(-\pi, \pi]$, so it isn't the principal argument of $3\sqrt{3} + 9i$. The principal argument is given by

$$-\frac{5\pi}{3} + 2\pi = -\frac{5\pi}{3} + \frac{6\pi}{3} = \frac{\pi}{3}.$$

Therefore

$$\begin{aligned} & (3\sqrt{3} + 9i)^{-5} \\ &= \frac{1}{(6\sqrt{3})^5} \left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right) \right) \\ &= \frac{1}{(6\sqrt{3})^5} \left(\frac{1}{2} + i \frac{\sqrt{3}}{2} \right) \\ &= \frac{1}{2(6\sqrt{3})^5} (1 + i\sqrt{3}) \\ &= \frac{1}{139\,968\sqrt{3}} (1 + i\sqrt{3}). \end{aligned}$$

Solution to Exercise 23

- (a) The equation is $z^2 + 2z + 3 = 0$.

Completing the square on the left-hand side gives

$$(z + 1)^2 - 1 + 3 = 0;$$

that is,

$$(z + 1)^2 = 1 - 3 = -2.$$

Taking square roots of both sides gives

$$z + 1 = \pm i\sqrt{2},$$

so

$$z = -1 \pm i\sqrt{2}.$$

- (b) The equation is $z^2 - 3z + 13 = 0$.

Completing the square on the left-hand side gives

$$\left(z - \frac{3}{2}\right)^2 - \frac{9}{4} + 13 = 0;$$

that is,

$$\left(z - \frac{3}{2}\right)^2 = \frac{9}{4} - 13 = -\frac{43}{4}.$$

Taking square roots of both sides gives

$$z - \frac{3}{2} = \pm i\sqrt{\frac{43}{4}},$$

so

$$z = \frac{3}{2} \pm \frac{\sqrt{43}}{2}i.$$

- (c) The equation is $z^2 + 36 = 0$.

Subtracting 36 from both sides gives

$$z^2 = -36.$$

Taking square roots of both sides gives

$$z = \pm 6i.$$

Solution to Exercise 24

$$(a) \quad z = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 1 \times 3}}{2 \times 1}$$

$$= \frac{2 \pm \sqrt{-8}}{2}$$

$$= \frac{2 \pm i\sqrt{8}}{2}$$

$$= \frac{2 \pm 2\sqrt{2}i}{2}$$

$$= 1 \pm i\sqrt{2}$$

$$(b) \quad z = \frac{-3 \pm \sqrt{3^2 - 4 \times 1 \times 13}}{2 \times 1}$$

$$= \frac{-3 \pm \sqrt{-43}}{2}$$

$$= \frac{-3 \pm i\sqrt{43}}{2}$$

$$(c) \quad z = \frac{-6 \pm \sqrt{6^2 - 4 \times 2 \times 5}}{2 \times 2}$$

$$= \frac{-6 \pm \sqrt{-4}}{4}$$

$$= \frac{-6 \pm 2i}{4}$$

$$= \frac{-3 \pm i}{2}$$

Solution to Exercise 25

- (a) A quadratic equation with solutions $2 \pm 3i$ is

$$(z - (2 + 3i))(z - (2 - 3i)) = 0.$$

Simplifying gives

$$((z - 2) - 3i)((z - 2) + 3i) = 0$$

$$(z - 2)^2 - (3i)^2 = 0$$

$$z^2 - 4z + 4 - (-9) = 0$$

$$z^2 - 4z + 13 = 0.$$

Exercise Booklet 12

- (b) A quadratic equation with solutions $-3 \pm 2i$ is

$$(z - (-3 + 2i))(z - (-3 - 2i)) = 0.$$

Simplifying gives

$$((z + 3) - 2i)((z + 3) + 2i) = 0$$

$$(z + 3)^2 - (2i)^2 = 0$$

$$z^2 + 6z + 9 - (-4) = 0$$

$$z^2 + 6z + 13 = 0.$$

- (c) A quadratic equation with solutions $4 \pm \frac{1}{2}i$ is

$$(z - (4 + \frac{1}{2}i))(z - (4 - \frac{1}{2}i)) = 0.$$

Simplifying gives

$$((z - 4) - \frac{1}{2}i)((z - 4) + \frac{1}{2}i) = 0$$

$$(z - 4)^2 - (\frac{1}{2}i)^2 = 0$$

$$z^2 - 8z + 16 - (-\frac{1}{4}) = 0$$

$$z^2 - 8z + \frac{65}{4} = 0.$$

- (d) A quadratic equation with solutions $\pm 6i$ is

$$(z - 6i)(z + 6i) = 0.$$

Simplifying gives

$$z^2 - (6i)^2 = 0$$

$$z^2 - (-36) = 0$$

$$z^2 + 36 = 0.$$

Solution to Exercise 26

Let $z = r(\cos \theta + i \sin \theta)$. Also,

$$1 = \cos 0 + i \sin 0.$$

So the equation $z^8 = 1$ in polar form is

$$(r(\cos \theta + i \sin \theta))^8 = \cos 0 + i \sin 0.$$

De Moivre's formula gives

$$r^8(\cos 8\theta + i \sin 8\theta) = \cos 0 + i \sin 0.$$

Comparing the moduli gives

$$r^8 = 1, \quad \text{so} \quad r = 1.$$

Comparing the arguments gives

$$8\theta = 0 + 2m\pi = 2m\pi,$$

where m is an integer. Hence

$$\theta = \frac{2m\pi}{8} = \frac{m\pi}{4},$$

where m is an integer.

Taking $m = 0, 1, 2, 3, 4, 5, 6, 7$ gives the following values of θ :

$$\theta = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, \pi, \frac{5\pi}{4}, \frac{3\pi}{2}, \frac{7\pi}{4}.$$

The solutions are therefore

$$z_0 = \cos 0 + i \sin 0 = 1$$

$$z_1 = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}(1 + i)$$

$$z_2 = \cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) = i$$

$$z_3 = \cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) = \frac{1}{\sqrt{2}}(-1 + i)$$

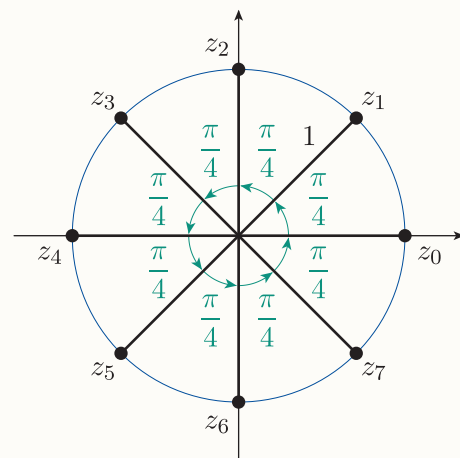
$$z_4 = \cos \pi + i \sin \pi = -1$$

$$z_5 = \cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}}(1 + i)$$

$$z_6 = \cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) = -i$$

$$z_7 = \cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}}(1 - i).$$

All other values of m give repetitions of these eight solutions.



(The eight solutions are evenly spaced around the unit circle, that is, the circle with radius 1 centred on the origin.)

Solution to Exercise 27

(a) Let $z = r(\cos \theta + i \sin \theta)$. Also,

$$-16 = 16(\cos \pi + i \sin \pi).$$

So the equation $z^4 = -16$ in polar form is

$$r^4(\cos 4\theta + i \sin 4\theta) = 16(\cos \pi + i \sin \pi).$$

Comparing the moduli gives

$$r^4 = 16, \quad \text{so} \quad r = 2.$$

Comparing the arguments gives

$$4\theta = \pi + 2m\pi, \quad \text{where } m \text{ is an integer.}$$

Hence

$$\theta = \frac{\pi}{4} + \frac{m\pi}{2}, \quad \text{where } m \text{ is an integer.}$$

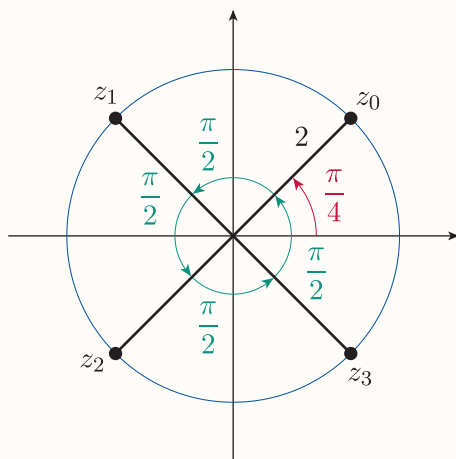
Taking $m = 0, 1, 2, 3$ gives the solutions in polar form as

$$\begin{aligned} z_0 &= 2 \left(\cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right) \right) \\ z_1 &= 2 \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right) \\ z_2 &= 2 \left(\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) \right) \\ z_3 &= 2 \left(\cos\left(\frac{7\pi}{4}\right) + i \sin\left(\frac{7\pi}{4}\right) \right). \end{aligned}$$

All other values of m give repetitions of these four solutions.

In Cartesian form these are

$$\begin{aligned} z_0 &= \sqrt{2}(1 + i), & z_1 &= \sqrt{2}(-1 + i), \\ z_2 &= -\sqrt{2}(1 + i), & z_3 &= \sqrt{2}(1 - i). \end{aligned}$$



(b) Let $z = r(\cos \theta + i \sin \theta)$. Also,

$$27i = 27 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right).$$

So the equation $z^3 = 27i$ in polar form is

$$\begin{aligned} r^3(\cos 3\theta + i \sin 3\theta) &= 27 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right). \end{aligned}$$

Comparing the moduli gives

$$r^3 = 27, \quad \text{so} \quad r = 3.$$

Comparing the arguments gives

$$3\theta = \frac{\pi}{2} + 2m\pi, \quad \text{where } m \text{ is an integer.}$$

Hence

$$\theta = \frac{\pi}{6} + \frac{2m\pi}{3}, \quad \text{where } m \text{ is an integer.}$$

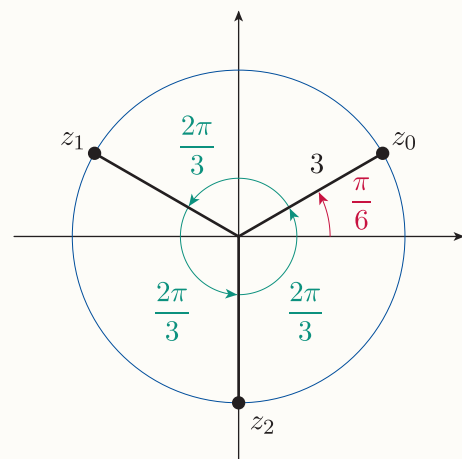
Taking $m = 0, 1, 2$ gives the solutions in polar form as

$$\begin{aligned} z_0 &= 3 \left(\cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) \right) \\ z_1 &= 3 \left(\cos\left(\frac{5\pi}{6}\right) + i \sin\left(\frac{5\pi}{6}\right) \right) \\ z_2 &= 3 \left(\cos\left(\frac{3\pi}{2}\right) + i \sin\left(\frac{3\pi}{2}\right) \right). \end{aligned}$$

All other values of m give repetitions of these three solutions.

In Cartesian form these are

$$\begin{aligned} z_0 &= \frac{3}{2}(\sqrt{3} + i), & z_1 &= \frac{3}{2}(-\sqrt{3} + i), \\ z_2 &= -3i. \end{aligned}$$

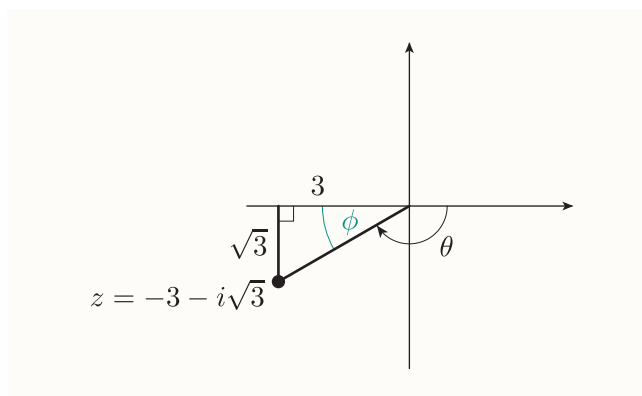


Solution to Exercise 28

- (a) $5e^{i\pi/2} = 5 \left(\cos\left(\frac{\pi}{2}\right) + i \sin\left(\frac{\pi}{2}\right) \right)$
 $= 5(0 + i)$
 $= 5i$
- (b) $2e^{-i\pi/6} = 2 \left(\cos\left(-\frac{\pi}{6}\right) + i \sin\left(-\frac{\pi}{6}\right) \right)$
 $= 2 \left(\frac{\sqrt{3}}{2} - \frac{i}{2} \right)$
 $= \sqrt{3} - i$
- (c) $2e^{3\pi i/4} = 2 \left(\cos\left(\frac{3\pi}{4}\right) + i \sin\left(\frac{3\pi}{4}\right) \right)$
 $= 2 \left(-\frac{1}{\sqrt{2}} + i \frac{1}{\sqrt{2}} \right)$
 $= \sqrt{2}(-1 + i)$

Solution to Exercise 29

- (a) The modulus is 3 and the principal argument is 0. Therefore
 $3 = 3e^{i0}$.
- (b) The modulus is $\frac{1}{2}$ and the principal argument is $\pi/2$. Therefore
 $\frac{1}{2}i = \frac{1}{2}e^{i\pi/2}$.
- (c) The modulus is $\frac{3}{2}$ and the principal argument is $-\pi/2$. Therefore
 $-\frac{3}{2}i = \frac{3}{2}e^{-i\pi/2}$.
- (d) The modulus of $-3 - i\sqrt{3}$ is
 $\sqrt{(-3)^2 + (-\sqrt{3})^2} = \sqrt{12} = 2\sqrt{3}$.



From the diagram,

$$\tan \phi = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}.$$

Therefore $\phi = \pi/6$. So the principal argument is

$$\theta = -(\pi - \phi) = -\left(\pi - \frac{\pi}{6}\right) = -\frac{5\pi}{6}.$$

Hence

$$-3 - i\sqrt{3} = 2\sqrt{3} e^{-5\pi i/6}.$$

Solution to Exercise 30

- (a) In exponential form,
 $e^{i\pi/6} \times e^{i\pi/2} = e^{i(\pi/6 + \pi/2)}$
 $= e^{i(4\pi/6)}$
 $= e^{2\pi i/3}$.
- In Cartesian form,
 $e^{2\pi i/3} = \cos\left(\frac{2\pi}{3}\right) + i \sin\left(\frac{2\pi}{3}\right)$
 $= \frac{1}{2}(-1 + i\sqrt{3})$.

- (b) In exponential form,
 $(e^{i\pi/8})^2 = e^{i(2 \times \pi/8)} = e^{i\pi/4}$.

In Cartesian form,

$$e^{i\pi/4} = \cos\left(\frac{\pi}{4}\right) + i \sin\left(\frac{\pi}{4}\right)$$

$$= \frac{1}{\sqrt{2}}(1 + i).$$

- (c) In exponential form,
 $\frac{e^{i\pi/2}}{e^{i\pi/3}} = e^{i(\pi/2 - \pi/3)} = e^{i\pi/6}$.

In Cartesian form,

$$e^{i\pi/6} = \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{2}(\sqrt{3} + i).$$

- (d) In exponential form,
 $2e^{i\pi/3} \times 3e^{-i\pi/3} = 2 \times 3e^{i(\pi/3 + (-\pi/3))}$
 $= 6e^{i0}$.

In Cartesian form,

$$6e^{i0} = 6.$$

- (e) In exponential form,
 $(4e^{i\pi})^3 = 4^3 e^{3\pi i} = 64e^{i\pi}$.

In Cartesian form,

$$64e^{i\pi} = 64(\cos \pi + i \sin \pi)$$

$$= 64(-1 + 0i)$$

$$= -64.$$

Solution to Exercise 31

Substituting $\theta = -\pi/2$ into Euler's formula gives

$$\begin{aligned} e^{-i\pi/2} &= \cos\left(-\frac{\pi}{2}\right) + i\sin\left(-\frac{\pi}{2}\right) \\ &= 0 - i. \end{aligned}$$

Hence $e^{-i\pi/2} = -i$, so $e^{-i\pi/2} + i = 0$.

Solution to Exercise 32

For brevity, write $c = \cos \theta$ and $s = \sin \theta$. Then, by de Moivre's formula and the binomial theorem,

$$\begin{aligned} \cos 5\theta + i \sin 5\theta &= (\cos \theta + i \sin \theta)^5 \\ &= (c + is)^5 \\ &= c^5 + 5c^4(is) + 10c^3(is)^2 + 10c^2(is)^3 \\ &\quad + 5c(is)^4 + (is)^5 \\ &= c^5 + 5ic^4s - 10c^3s^2 - 10ic^2s^3 + 5cs^4 + is^5 \\ &= (c^5 - 10c^3s^2 + 5cs^4) + i(5c^4s - 10c^2s^3 + s^5). \end{aligned}$$

Comparing real and imaginary parts, and using the identity $c^2 + s^2 = 1$, gives

$$\begin{aligned} \cos 5\theta &= c^5 - 10c^3s^2 + 5cs^4 \\ &= c^5 - 10c^3(1 - c^2) + 5c(1 - c^2)(1 - c^2) \\ &= c^5 - 10c^3 + 10c^5 + 5c(1 - 2c^2 + c^4) \\ &= c^5 - 10c^3 + 10c^5 + 5c - 10c^3 + 5c^5 \\ &= 5c - 20c^3 + 16c^5 \\ &= 5 \cos \theta - 20 \cos^3 \theta + 16 \cos^5 \theta \end{aligned}$$

and

$$\begin{aligned} \sin 5\theta &= 5c^4s - 10c^2s^3 + s^5 \\ &= 5(1 - s^2)(1 - s^2)s - 10(1 - s^2)s^3 + s^5 \\ &= 5(1 - 2s^2 + s^4)s - 10s^3 + 10s^5 + s^5 \\ &= 5s - 10s^3 + 5s^5 - 10s^3 + 10s^5 + s^5 \\ &= 5s - 20s^3 + 16s^5 \\ &= 5 \sin \theta - 20 \sin^3 \theta + 16 \sin^5 \theta, \end{aligned}$$

which are the required identities.

Solution to Exercise 33

(a) We have

$$\sin \theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \quad \text{and} \quad \cos \theta = \frac{e^{i\theta} + e^{-i\theta}}{2}.$$

Therefore

$$\begin{aligned} \sin^2 \theta + \cos^2 \theta &= \frac{(e^{i\theta} - e^{-i\theta})^2}{(2i)^2} + \frac{(e^{i\theta} + e^{-i\theta})^2}{2^2} \\ &= -\frac{1}{4}((e^{i\theta})^2 - 2e^{i\theta}e^{-i\theta} + (e^{-i\theta})^2) \\ &\quad + \frac{1}{4}((e^{i\theta})^2 + 2e^{i\theta}e^{-i\theta} + (e^{-i\theta})^2) \\ &= -\frac{1}{4}(e^{2i\theta} - 2e^{i(\theta+(-\theta))} + e^{-2i\theta}) \\ &\quad + \frac{1}{4}(e^{2i\theta} + 2e^{i(\theta+(-\theta))} + e^{-2i\theta}) \\ &= \frac{1}{4}(-e^{2i\theta} + e^{2i\theta} + 2e^0 + 2e^0 - e^{-2i\theta} + e^{-2i\theta}) \\ &= \frac{1}{4}(4e^0) \\ &= 1. \end{aligned}$$

(b) Similarly,

$$\begin{aligned} \cos^2 \theta - \sin^2 \theta &= \frac{(e^{i\theta} + e^{-i\theta})^2}{2^2} - \frac{(e^{i\theta} - e^{-i\theta})^2}{(2i)^2} \\ &= \frac{1}{4}((e^{i\theta})^2 + 2e^{i\theta}e^{-i\theta} + (e^{-i\theta})^2) \\ &\quad + \frac{1}{4}((e^{i\theta})^2 - 2e^{i\theta}e^{-i\theta} + (e^{-i\theta})^2) \\ &= \frac{1}{4}(e^{2i\theta} + 2e^{i(\theta+(-\theta))} + e^{-2i\theta}) \\ &\quad + \frac{1}{4}(e^{2i\theta} - 2e^{i(\theta+(-\theta))} + e^{-2i\theta}) \\ &= \frac{1}{4}(2e^{2i\theta} + 2e^{-2i\theta}) \\ &= \frac{(e^{2i\theta} + e^{-2i\theta})}{2} \\ &= \cos 2\theta. \end{aligned}$$

$$\begin{aligned} \text{(c) } 2 \sin \theta \cos \theta &= 2 \times \frac{e^{i\theta} - e^{-i\theta}}{2i} \times \frac{e^{i\theta} + e^{-i\theta}}{2} \\ &= 2 \times \frac{(e^{i\theta} - e^{-i\theta})(e^{i\theta} + e^{-i\theta})}{4i} \\ &= \frac{((e^{i\theta})^2 - (e^{-i\theta})^2)}{2i} \\ &= \frac{(e^{2i\theta} - e^{-2i\theta})}{2i} \\ &= \sin(2\theta). \end{aligned}$$